SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2005
HIGHER SCHOOL CERTIFICATE
TRIAL PAPER

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 3 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 120

- Attempt questions 1 – 8

Examiner: *C.Kourtesis*

NOTE: This is a trial paper only and does not necessarily reflect the content or format of the final Higher School Certificate examination paper for this subject.

Section A
(Start a new answer sheet.)

Question 1. (15 marks)

Marks

(a) Evaluate $\int_0^2 \frac{3}{4+x^2}\,dx$. **2**

(b) Find $\int \cos x \sin^4 x \, dx$. **1**

(c) Use integration by parts to find **2**

$$\int te^{-t}\,dt.$$

(d) (i) Find real numbers a and b such that **2**

$$\frac{1}{x(\pi-2x)} = \frac{a}{x} + \frac{b}{\pi-2x}.$$

(ii) Hence find **2**

$$\int \frac{dx}{x(\pi-2x)}.$$

(e) Evaluate $\int_{-3}^{3} \left(2-|x|\right)dx$. **2**

(f) (i) Use the substitution $x = a - t$ to prove that **2**

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx.$$

(ii) Hence evaluate **2**

$$\int_0^{\frac{\pi}{2}} \log_e\left(\tan x\right)dx$$

Question 2. (15 marks)

Marks

(a) If $z = 2 + i$ and $w = -1 + 2i$ find **2**

$$\text{Im}(z - w).$$

(b) On an Argand diagram shade the region that is satisfied by both the conditions **2**

$$\text{Re}(z) \geq 2 \text{ and } |z - 1| \leq 2 .$$

(c) If $|z| = 2$ and $\arg z = \theta$ determine **3**

(i) $\left|\dfrac{i}{z^2}\right|$ (ii) $\arg\left(\dfrac{i}{z^2}\right)$

(d) If for a complex number z it is given that $\overline{z} = z$ where $z \neq 0$, determine the locus of z. **2**

(e) A complex number z is such that $\arg(z + 2) = \dfrac{\pi}{6}$ and $\arg(z - 2) = \dfrac{2\pi}{3}$. **3**

Find z, expressing your answer in the form $a + ib$ where *a* and *b* are real.

(f) The complex numbers z_1, z_2 and z_3 are represented in the complex plane by the points *P*, *Q* and *R* respectively. If the line segments *PQ* and *PR* have the same length and are perpendicular to one another, prove that: **3**

$$2z_1^2 + z_2^2 + z_3^2 = 2z_1(z_2 + z_3)$$

Section B
(Start a new answer sheet.)

Question 3. (15 marks)

Marks

(a) If $2-3i$ is a zero of the polynomial z^3+pz+q where p and q are real, find the values of p and q. **3**

(b) If α, β and γ are roots of the equation $x^3+6x+1=0$ find the polynomial equation whose roots are $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$. **2**

(c) Consider the function $f(x)=3\left(\dfrac{x+4}{x}\right)^2$.

(i) Show that the curve $y=f(x)$ has a minimum turning point at $x=-4$ and a point of inflexion at $x=-6$. **5**

(ii) Sketch the graph of $y=f(x)$ showing clearly the equations of any asymptotes. **2**

(d) Use mathematical induction to prove that **3**

$$n!>2^n \text{ for } n>3 \text{ where } n \text{ is an integer.}$$

Question 4 (15 marks)

(a) If $f(x) = \sin x$ for $-\pi \le x \le \pi$ draw neat sketches, on separate diagrams, of:

(i) $y = \left[f(x) \right]^2$ **2**

(ii) $y = \dfrac{1}{f\left(x + \dfrac{\pi}{2}\right)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = f\left(\sqrt{|x|}\right)$ **2**

(b) Show that the equation of the tangent to the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ at the point $P\left(x_0, y_0\right)$ on the curve is $xx_o^{-\frac{1}{2}} + yy_0^{-\frac{1}{2}} = a^{\frac{1}{2}}$. **3**

(c) Consider the polynomial $P(x) = x^5 - ax + 1$. By considering turning points on the curve $y = P(x)$, prove that $P(x) = 0$ has three distinct roots if $a > 5\left(\dfrac{1}{2}\right)^{\frac{8}{5}}$. **4**

Section C
(Start a new answer booklet)

Question 5 (15 marks)

Marks

(a) A particle of mass m is thrown vertically upward from the origin with initial speed V_0. The particle is subject to a resistance equal to mkv, where v is its speed and k is a positive constant.

(i) Show that until the particle reaches its highest point the equation of motion is **1**

$$\ddot{y} = -(kv + g)$$

where y is its height and g is the acceleration due to gravity.

(ii) Prove that the particle reaches its greatest height in time T given by **4**

$$kT = \log_e \left[1 + \frac{kV_0}{g} \right].$$

(iv) If the highest point reached is at a height H above the ground prove that **4**

$$V_0 = Hk + gT\ .$$

(b) If α and β are roots of the equation $z^2 - 2z + 2 = 0$

(i) find α and β in mod-arg form. **3**

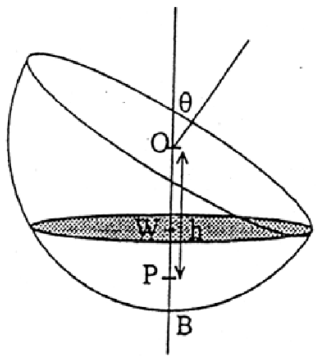
(ii) show that $\alpha^n + \beta^n = \sqrt{2^{n+2}} . \left[\cos\dfrac{n\pi}{4} \right]$. **3**

Question 6 (15 marks)

(a) A group of 20 people is to be seated at a long rectangular table, 10 on each side. There are 7 people who wish to sit on one side of the table and 6 people who wish to sit on the other side. How many seating arrangements are possible? **2**

(b) The area enclosed by the curves $y = \sqrt{x}$ and $y = x^2$ is rotated about the y axis through one complete revolution. Use the cylindrical shell method to find the volume of the solid that is generated. **3**

(c) The diagram shows a hemi-spherical bowl of radius r. The bowl has been tilted so that its axis is no longer vertical, but at an angle θ to the vertical. At this angle it can hold a volume V of water.



The vertical line from the centre O meets the surface of the water at W and meets the bottom of the bowl at B. Let P between W and B, and let h be the distance OP.

(i) Explain why $V = \int_{r\sin\theta}^{r} \pi\left(r^2 - h^2\right)dh$. **3**

(ii) Hence show $V = \dfrac{r^3\pi}{3}\left(2 - 3\sin\theta + \sin^3\theta\right)$. **2**

(d) (i) Show that $x^4 + y^4 \geq 2x^2y^2$. **2**

(ii) If $P(x, y)$ is any point on the curve $x^4 + y^4 = 1$ prove that $OP \leq 2^{\frac{1}{4}}$, where O is the origin. **3**

Section D
(Start a new answer booklet)

Question 7 (15 marks)

(a) How many sets of 5 quartets (groups of four musicians) can be formed from 5 violinists, 5 viola players, 5 cellists, and 5 pianists if each quartet is to consist of one player of each instrument? **2**

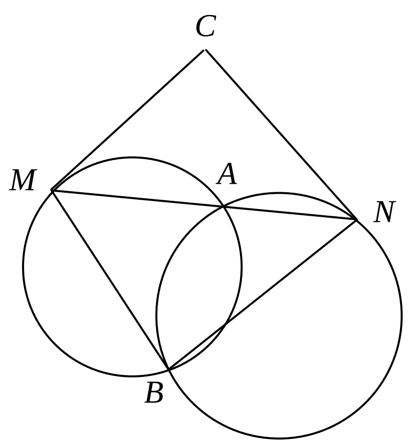
(b) (i) If $t = \tan\theta$, prove that **2**

$$\tan 4\theta = \frac{4t\left(1-t^2\right)}{1-6t^2+t^4}.$$

(ii) If $\tan\theta\tan 4\theta = 1$ deduce that $5t^4 - 10t^2 + 1 = 0$. **2**

(iii) Given that $\theta = \frac{\pi}{10}$ and $\theta = \frac{3\pi}{10}$ are roots of the equation $\tan\theta\tan 4\theta = 1$, find the exact value of $\tan\frac{\pi}{10}$. **4**

(c)



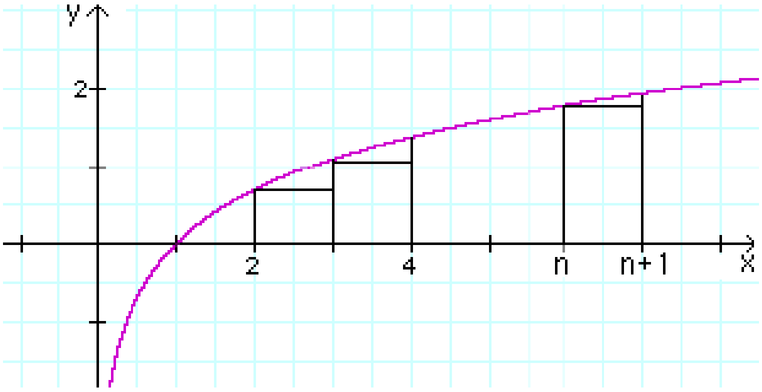
Two circles intersect at *A* and *B*. A line through *A* cuts the circles at *M* and *N*. The tangents at *M* and *N* intersect at *C*. **5**

(i) Prove that $\angle CMA + \angle CNA = \angle MBN$.

(ii) Prove *M*, *C*, *N*, *B* are concyclic.

Question 8 (15 marks)

(a) **6**



The diagram above shows the graph of $y = \log_e x$ for $1 \le x \le n+1$.

(i) By considering the sum of the areas of inner and outer rectangles show that

$$\ln(n!) < \int_1^{n+1} \ln x \, dx < \ln\left[(n+1)!\right]$$

(ii) Find $\int_1^{n+1} \ln x \, dx$.

(iii) Hence prove that

$$e^n > \frac{(n+1)^n}{n!}$$

(b) If a root of the cubic equation $x^3 + bx^2 + cx + d = 0$ is equal to the reciprocal of another root, prove that **3**

$$1 + bd = c + d^2.$$

This question continues on the next page.

(c) A stone is projected from a point O on a horizontal plane at an angle of elevation α and with initial velocity U metres per second. The stone reaches a point A in its trajectory, and at that instant it is moving in a direction perpendicular to the angle of projection with speed V metres per second. **6**

Air resistance is neglected throughout the motion and g is the acceleration due to gravity.

If t is the time in seconds at any instant, show that when the stone is at A:

(i) $V = U\cot\alpha$

(ii) $t = \dfrac{U}{g\sin\alpha}$.

This is the end of the paper.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \; n \neq -1; \; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \;\; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \;\; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right) \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \;\; x > 0$

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

AUGUST 2005

Trial Higher School Certificate
Examination

YEAR 12

Mathematics Extension 2

Sample Solutions

Section	Marker
A	PP
B	EC
C	PB
D	DH

Section A

(1) (i) $$\int_0^2 \frac{3}{4+x^2}\,dx = \frac{3}{2}\int_0^2 \frac{3}{4+x^2}\,dx$$

$$= \frac{3}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \frac{3}{2}\left[\tan^{-1}(1) - 0\right]$$

$$= \frac{3\pi}{8}$$

(ii) $$\int \cos x \sin^4 x dx = \int u^4 du \quad [u = \sin x]$$

$$= \frac{u^5}{5} + c$$

$$= \frac{\sin^5 x}{5} + c$$

(iii) $$\int \underset{g}{t}\,\underset{f'}{e^{-t}}\,dt = fg - \int fg' dt$$

$$= -te^{-t} - \int \left(-e^{-t} \times 1\right) dt$$

$$= -te^{-t} + \int e^{-t} dt$$

$$= -te^{-t} - e^{-t} + c$$

(d) (i) $1 \equiv a(\pi - 2x) + bx$

$x = 0 :\Rightarrow a = \frac{1}{\pi}$

$2a = b$ [coefficients of x]

$\therefore b = \frac{2}{\pi}$

$a = \frac{1}{\pi}, b = \frac{2}{\pi}$

(ii) $$\int \frac{dx}{x(\pi - 2x)} = \frac{1}{\pi}\int \left(\frac{1}{x} - \frac{-2}{\pi - 2x}\right) dx$$

$$= \frac{1}{\pi}\ln x - \frac{1}{\pi}\ln(\pi - 2x) + c$$

$$= \frac{1}{\pi}\ln\left(\frac{x}{\pi - 2x}\right) + c$$

(e) $\int_{-3}^{3}(2-|x|)dx = 2\int_{0}^{3}(2-|x|)dx$ $\quad [\because 2-|x| \text{ is even}]$

$= 2\int_{0}^{3}(2-x)dx$ $\quad [\because 2-|x| = 2-x, x>0]$

$= 2\left[2x-\frac{1}{2}x^2\right]_0^3$

$= 2\left[6-\frac{9}{2}\right]$

$= 3$

(f) (i) $x = a-t \Rightarrow dx = -dt$

$x = 0 \Rightarrow t = a$

$x = a \Rightarrow t = 0$

$\int_0^a f(x)dx = \int_a^0 f(a-t)(-dt)$

$= \int_0^a f(a-t)dt$ $\quad \left[\because \int_a^b f(x)dx = -\int_b^a f(x)dx\right]$

$= \int_0^a f(a-x)dx$ $\quad [\because \text{choice of variable irrelevant}]$

(ii) $I = \int_0^{\frac{\pi}{2}} \ln(\tan x)dx$

$= \int_0^{\frac{\pi}{2}} \ln\left(\tan\left(\frac{\pi}{2}-x\right)\right)dx$

$= \int_0^{\frac{\pi}{2}} \ln(\cot x)dx$

$2I = \int_0^{\frac{\pi}{2}} \ln(\tan x)dx + \int_0^{\frac{\pi}{2}} \ln(\cot x)dx$

$= \int_0^{\frac{\pi}{2}} \left[\ln(\tan x) + \ln(\cot x)\right]dx$

$= \int_0^{\frac{\pi}{2}} \ln 1 dx$

$= 0$

$\therefore 2I = 0$

$\therefore I = 0$

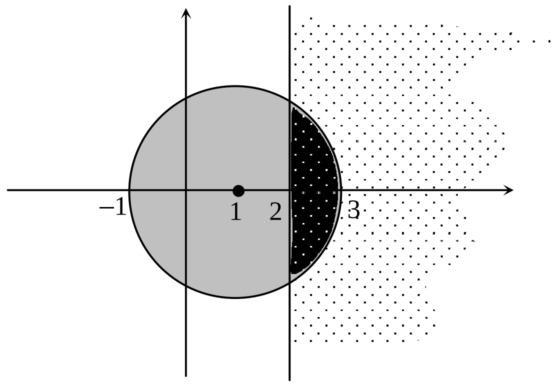
$\therefore \int_0^{\frac{\pi}{2}} \ln(\tan x)dx = 0$

(2) (a) $z = 2 + i, w = -1 + 2i$

$\therefore z - w = 3 - i$

$\therefore \text{Im}(z - w) = -1$

(b)



(c) (i) $\left|\frac{i}{z^2}\right| = \frac{|i|}{|z|^2} = \frac{1}{4}$

(ii) $\arg\left(\frac{i}{z^2}\right) = \arg i - \arg\left(z^2\right)$

$$= \frac{\pi}{2} - 2\arg z$$

$$= \frac{\pi}{2} - 2\theta$$

(d) $z = \overline{z} \Rightarrow z$ is purely real

So the locus is $y = 0$, except $x = 0$.

Alternatively:

Let $z = x + iy, (z \neq 0)$

$\therefore \overline{z} = x - iy$

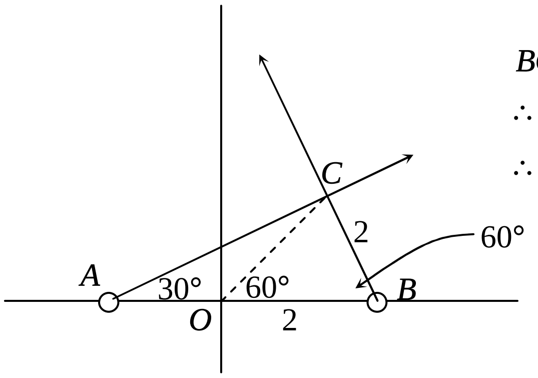
$\therefore z = \overline{z} \Rightarrow x + iy = x - iy$

$\therefore 2iy = 0 \Rightarrow y = 0$

$\therefore z$ is a purely real number excluding 0

(e) $\arg(z+2)=\dfrac{\pi}{6}, \arg(z-2)=\dfrac{2\pi}{3}$.

z is represented by the point C, the intersection of the two rays.

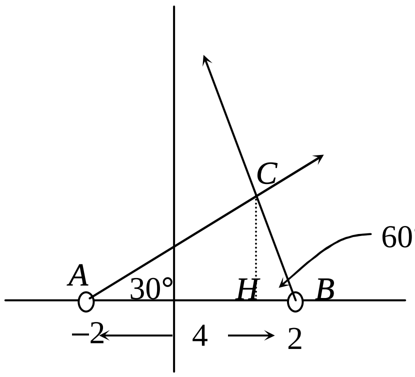


$BC = 4\cos 60° = 2$

$\therefore \Delta BOC$ is equilateral

$\therefore z = 2\text{cis}60° = 1 + i\sqrt{3}$

Alternatively



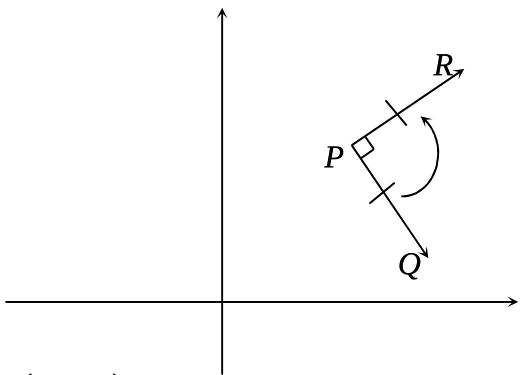
$BC = 4\cos 60° = 2$

$CH = 2\sin 60° = \sqrt{3}$

$BH = 2\cos 60° = 1$

$\therefore z = 1 + i\sqrt{3}$

(f)



$\overrightarrow{PR} \perp \overrightarrow{PQ}$

$\left|\overrightarrow{PR}\right| = \left|\overrightarrow{PQ}\right|$

$\overrightarrow{PR}$ represents $z_3 - z_1$

$\overrightarrow{PQ}$ represents $z_2 - z_1$

$i(z_2 - z_1) = z_3 - z_1$

$\therefore i^2 (z_2 - z_1)^2 = (z_3 - z_1)^2$

$\therefore -(z_2{}^2 - 2z_2 z_1 + z_1{}^2) = z_3{}^2 - 2z_1 z_3 + z_1{}^2$

$\therefore 2z_1{}^2 + z_2{}^2 + z_3{}^2 = 2z_1 z_3 + 2z_2 z_1 = 2z_1(z_2 + z_3)$

Section B

Question (3)

(a) $z^3 + pz + q = 0$

If $(2-3i)$ is a zero then

$(2-3i)^3 + (2-3i)p + q = 0$

$(2-3i)^3 = (2-3i)^2(2-3i)$

$= (-5-12i)(2-3i)$

$= -46 - 9i$ 1

$\therefore -46 - 9i + (2-3i)p + q = 0$

Equating real parts.

$-46 + 2p + q = 0$ 1

$2p + q = 46$ —— (1)

Equating imaginary parts.

$-9 - 3p = 0 \Rightarrow 3p = -9$

$\therefore \boxed{p = -3}$ —— (2)

Subst. (2) into (1)

We have 1

$-6 + q = 46$

$\therefore \boxed{q = 52}$ —— (3)

[3]

(b) $x^3 + 6x + 1 = 0$

If α, β, γ are the roots then $\alpha\beta\gamma = -1$

Now, $\alpha\beta = \frac{\alpha\beta\gamma}{\gamma} = -\frac{1}{\gamma}$

Similarly, $\beta\gamma = -\frac{1}{\alpha}$ 1

and $\alpha\gamma = -\frac{1}{\beta}$.

$\therefore$ Let $y = -\frac{1}{x} \Rightarrow x = -\frac{1}{y}$

i.e the polynomial equation is $\left(-\frac{1}{y}\right)^3 + 6\left(-\frac{1}{y}\right) + 1 = 0$. 1

$-1 - 6y^2 + y^3 = 0$

$\therefore \boxed{y^3 - 6y^2 - 1 = 0}$ [2]

(c) $y = 3\left(\frac{x+4}{x}\right)^2$.

$\frac{dy}{dx} = 6\left(\frac{x+4}{x}\right)\left(\frac{-4}{x^2}\right)$

$= \frac{-24(x+4)}{x^3}$

$\frac{dy}{dx} = 0 \Rightarrow x = -4$ 2

When $x = -4$, $y = 0$.

$\therefore (-4, 0)$ is a stationary pt.

$\frac{d^2y}{dx^2} = -24\left[\frac{x^3 - (x+4)3x^2}{x^6}\right]$

$= 24\left[\frac{2x^3 + 12x^2}{x^6}\right]$

$= 48x^2\left(\frac{x+6}{x^4}\right)$

$f''(x) = 0, \Rightarrow x = -6$.

When $x = -6$, $y = 3\left(\frac{4}{36}\right) = \frac{1}{3}$ 2

$\therefore (-6, \frac{1}{3})$ could be a pt of inflexion.

Test:

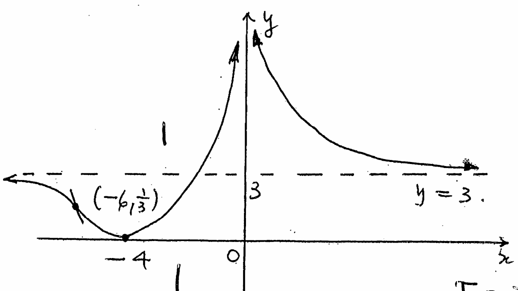
x	-7	-6	-5
$f''(x)$	$-$	0	$+$

$\therefore (-6, \frac{1}{3})$ is a pt. of inflexion.

Also, $f''(-4) = \frac{48}{4^4}(2) > 0$

1 $\therefore (-4, 0)$ is a min turning pt

$x \neq 0 \therefore$ y-axis is a vertical asymptote. [5]



[2].

Note:

$\lim_{x \to \pm\infty} 3\left(\frac{x^2 + 8x + 16}{x^2}\right) = 3$.

i.e horizontal asymptote at $y = 3$.

(d) Let $S(n)$ be the proposition that $n! > 2^n$ $n > 3$. $n \in Z^+$.

For $n = 4$

$4! = 24 > 2^4 = 16$

$\therefore$ $S(4)$ is true

Assume $S(k)$ is true.

Prove true for $n = k+1$.

$$(k+1)! = (k+1)k!$$
$$> (k+1) \cdot 2^k$$

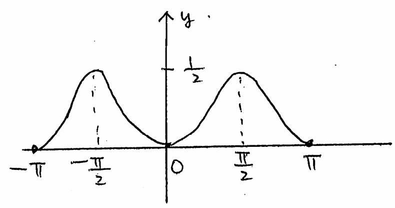
$\because$ $k > 3$, then $k+1 > 4$

$$> 4 \cdot 2^k$$
$$2 \cdot 2^{k+1}$$
$$> 2^{k+1}$$

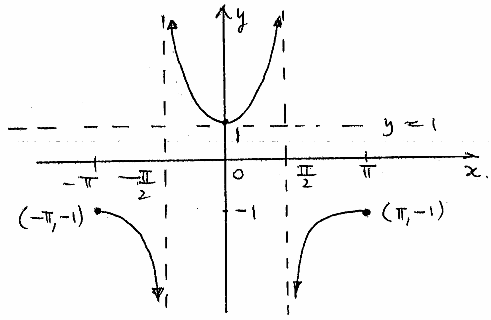
Question (4).

$y = \sin x$, $\quad -\pi \le x \le \pi$

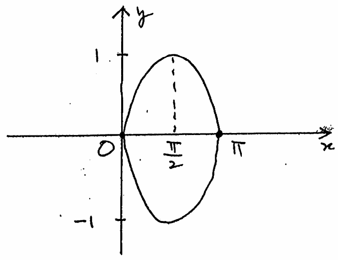
(i) $\quad y = \sin^2 x$
$$= \tfrac{1}{2}(1 - \cos 2x)$$



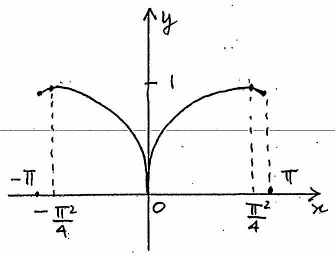
(ii) $\quad y = \dfrac{1}{\sin(x+\frac{\pi}{2})} = \dfrac{1}{\cos x}$.



(iii)
$$y^2 = \sin x$$
$$y = \pm\sqrt{\sin x}.$$



(iv)



(b) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$

$\therefore$ $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2} \cdot \frac{dy}{dx} = 0$.

$\therefore$ $\dfrac{dy}{dx} = -\dfrac{x^{-1/2}}{y^{-1/2}} = -\sqrt{\dfrac{y}{x}}$.

$$\left.\frac{dy}{dx}\right|_{\substack{x=x_0\\ y=y_0}} = -\sqrt{\frac{y_0}{x_0}}.$$

Equation of tgt.

$$y - y_0 = -\frac{y_0^{1/2}}{x_0^{1/2}}(x - x_0)$$

$\therefore$ $x_0^{\frac{1}{2}}y - x_0^{1/2}y_0 = x_0 y_0^{\frac{1}{2}} - x y_0^{\frac{1}{2}}$.

$\Longrightarrow$ $y_0^{\frac{1}{2}}x + x_0^{\frac{1}{2}}y = x_0^{\frac{1}{2}}y_0^{\frac{1}{2}}(y_0^{\frac{1}{2}} + x_0^{\frac{1}{2}})$ ①

$\because$ $(x_0\, y_0)$ is on the curve

$\Longrightarrow$ $x_0^{\frac{1}{2}} + y_0^{\frac{1}{2}} = a^{\frac{1}{2}}$

and divide both sides of ① by $x_0^{\frac{1}{2}}\, y_0^{\frac{1}{2}}$ we have.

$$x_0^{-1/2}x + y_0^{-1/2}y = a^{\frac{1}{2}}$$

(c)

$$p(x) = x^5 - ax + 1$$
$$p'(x) = 5x^4 - a$$
$$p''(x) = 20x^3$$

Stationary pts when

$$5x^4 - a = 0$$
$$x^4 = \frac{a}{5}$$
$$\therefore \quad x = \pm\left(\frac{a}{5}\right)^{\frac{1}{4}} \qquad 1$$

For $x = \left(\frac{a}{5}\right)^{\frac{1}{4}}$, $p''\left[\left(\frac{a}{5}\right)^{\frac{1}{4}}\right] > 0$

For $x = -\left(\frac{a}{5}\right)^{\frac{1}{4}}$, $p''\left[-\left(\frac{a}{5}\right)^{\frac{1}{4}}\right] < 0$

 1

When $x = \left(\frac{a}{5}\right)^{\frac{1}{4}}$, $y = \left(\frac{a}{5}\right)^{5/4} - a\left(\frac{a}{5}\right)^{\frac{1}{4}} + 1$

$$\therefore \quad y = 1 - \left(\frac{4a}{5}\right)\left(\frac{a}{5}\right)^{\frac{1}{5}}$$

For $x = -\left(\frac{a}{5}\right)^{\frac{1}{4}}$

$$y = 1 + \left(\frac{4a}{5}\right)\left(\frac{a}{5}\right)^{\frac{1}{2}}$$

For three distinct real roots, the curve cuts the x-axis in 3 points.

$x \longrightarrow \infty \quad p(x) \longrightarrow \infty \ (>0)$

and the turning points are on the OPPOSITE sides of the x-axis

$\Rightarrow$ The product of the y's < 0

$$\therefore \left[1 + \left(\frac{4a}{5}\right)\left(\frac{a}{5}\right)^{\frac{1}{4}}\right]\left[\left(1 - \left(\frac{4a}{5}\right)\left(\frac{a}{5}\right)^{\frac{1}{4}}\right] < 0$$

$$1 - \frac{16a^2}{25}\left(\frac{a}{5}\right)^{\frac{1}{2}} < 0 \qquad 1$$

$$\frac{16a^{5/2}}{5^{5/2}} > 1 \Rightarrow a^{5/2} > \left(\frac{5^{\frac{5}{2}}}{2^4}\right) \qquad 1$$

$$\therefore \quad a > \left(\frac{5^{5/2}}{2^4}\right)^{2/5} = \left[\frac{5^1}{2^{8/5}}\right]$$

$$\Rightarrow \quad a > 5\left(\frac{1}{2}\right)^{\frac{8}{5}} \qquad [4]$$

Section C

QUESTIONS

(a) (i) $m\ddot{y} = -mkv - mg$ $\qquad \uparrow m\ddot{y} \quad \downarrow mg \quad \downarrow mkv$

$$\therefore \boxed{\ddot{y} = -(kv + g).}$$

(ii) $\ddot{y} = \frac{dv}{dt} = -(kv+g)$

$$\therefore dt = -(kv+g)^{-1}\, dv$$

$$\int_0^T 1 . dt = - \int_{V_0}^{0} (kv+g)^{-1} dv.$$

$$T = \int_0^{V_0} (kv+g)^{-1} dv.$$

$$= \int_0^{V_0} \frac{1}{kv+g}\, dv.$$

$$= \frac{1}{k}\Big[\ln(kv+g)\Big]_0^{V_0}$$

$$= \frac{1}{k}\left(\ln(kV_0+g) - \ln g\right)$$

$$= \frac{1}{k}\ln\left(\frac{kV_0+g}{g}\right)$$

$$\therefore \boxed{kT = \ln\left(1 + \frac{kV_0}{g}\right)}$$

(iii) $$v \cdot \frac{dv}{dy} = -(kv + g)$$

$$\therefore \frac{dv}{dy} = -\left(\frac{kv+g}{v}\right)$$

$$dy = \frac{-v \cdot dv}{kv+g}$$

$$= -\frac{1}{k}\left(\frac{kv+g-g}{kv+g}\right) dv$$

$$\int_0^H 1 \cdot dy = -\frac{1}{k}\int_{V_0}^{0} \frac{kv+g-g}{kv+g}\, dv$$

$$\therefore H = \frac{1}{k}\int_0^{V_0}\left(1 - \frac{g}{kv+g}\right) dv$$

$$= \frac{1}{k}\left[\int_0^{V_0} 1 \cdot dv - \int_0^{V_0} \frac{g}{kv+g}\, dv\right]$$

$$= \frac{1}{k}\left[\left[v\right]_0^{V_0} - \left[\frac{g}{k}\ln(kv+g)\right]_0^{V_0}\right]$$

$$= \frac{1}{k}\left[V_0 - \frac{g}{k}\left(\ln(kV_0+g) - \ln g\right)\right]$$

$$kH = V_0 - \frac{g}{k}\ln\frac{kV_0+g}{g}$$

$$= V_0 - \frac{g}{k}kT \quad \text{from (ii)}$$

$$= V_0 - gT$$

$$\therefore \boxed{V_0 = kH + gT}$$

(b) (i) $z = \frac{2 \pm \sqrt{4-8}}{2}$

$= \frac{2 \pm 2i}{2}$

$= 1 \pm i$

$= \sqrt{2} \operatorname{cis} \pm \frac{\pi}{4}$

$\therefore \alpha, \beta$ are $\sqrt{2} \operatorname{cis} \pm \frac{\pi}{4}$

(ii) $\alpha^n + \beta^n = \left(\sqrt{2} \operatorname{cis} \frac{\pi}{4}\right)^n + \left(\sqrt{2} \operatorname{cis} \frac{-\pi}{4}\right)^n$

$= (\sqrt{2})^n \left[\operatorname{cis} \frac{n\pi}{4} + \operatorname{cis} -\frac{n\pi}{4}\right]$

$= (\sqrt{2})^n \left(2 \cos \frac{n\pi}{4}\right)$ (NB $z + \bar{z} = 2 \operatorname{Re} z$)

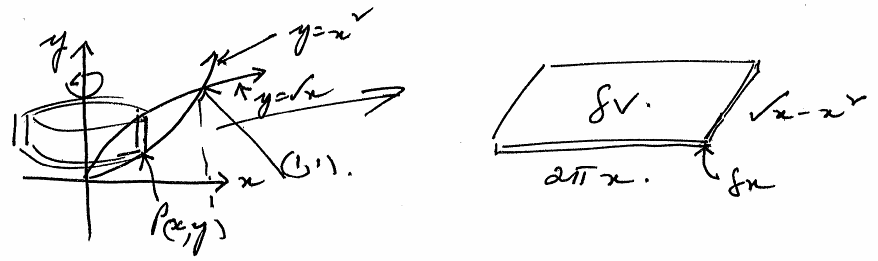
$= 2^{\frac{1}{2}n+1} \cos \frac{n\pi}{4}$

$= 2^{\frac{n+2}{2}} \cos \frac{n\pi}{4}$

$\therefore \alpha^n + \beta^n = \sqrt{2^{n+2}} \cos \frac{n\pi}{4}.$

QUESTION 6.

a $\boxed{{}^{10}P_7 \times {}^{10}P_6 \times 7!}$ (this is the same as $\binom{10}{7} \times \binom{10}{6} \times (7!)^2 \times 6!$)

b 

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi x\left(\sqrt{x} - x^2\right)\delta x.$$

$$= 2\pi \int_0^1 \left(x^{3/2} - x^3\right) dx.$$

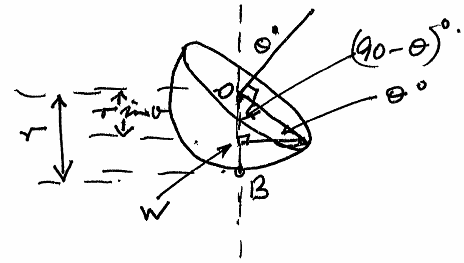
$$= 2\pi \cdot \left[\frac{2}{5}x^{5/2} - \frac{x^4}{4}\right]_0^1$$

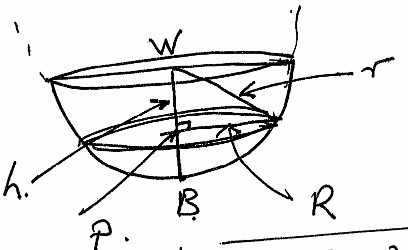
$$= 2\pi \left(\frac{2}{5} - \frac{1}{4}\right)$$

$$= 2\pi \times \frac{3}{20}$$

$$\boxed{= \frac{3\pi}{10} \ u^3}$$

(c) (i)





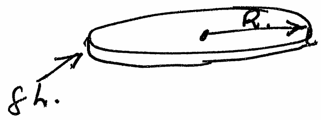
now $R^2 = r^2 - h^2$

NB $\frac{OW}{r} = \sin\theta$

$OW = r\sin\theta$

and $OB = r$

Consider the slice at P.



$$\delta V = \pi R^2 . \delta h$$

$$\therefore V = \lim_{\delta h \to 0} \sum_{h = r\sin\theta}^{h = r} . \pi R^2 \delta h$$

$$= \pi \int_{r\sin\theta}^{r} R^2 . dh$$

$$= \pi \int_{r\sin\theta}^{r} (r^2 - h^2) . dh$$

(ii) $$V = \pi \left[r^2 h - \frac{h^3}{3} \right]_{r\sin\theta}^{r}$$

$$= \pi \left[r^3 - \frac{r^3}{3} - \left(r^3 \sin\theta - \frac{r^3 \sin^3\theta}{3} \right) \right]$$

$$= \pi \left[\frac{2r^3}{3} - r^3 \sin\theta + \frac{r^3 \sin^3\theta}{3} \right]$$

$$= \frac{\pi r^3}{3} (2 - 3\sin\theta + \sin^3\theta)$$

(d) (i) now $(x^2-y^2)^2 \geq 0$.

$$x^4-2x^2y^2+y^4 \geq 0$$

$$\therefore \boxed{x^4+y^4 \geq 2x^2y^2}$$

(ii) now $OP = \sqrt{x^2+y^2}$

$$\therefore OP^4 = (x^2+y^2)^2$$

$$= x^4+y^4+2x^2y^2$$

$$\leq x^4+y^4+x^4+y^4 \quad (\text{from (i)})$$

$$\leq 2. \quad (x^4+y^4=1)$$

$$\therefore \boxed{OP \leq 2^{\frac{1}{4}}}$$

Section D

7. (a) How many sets of 5 quartets (groups of four musicians) can be formed from 5 violinists, 5 viola players, 5 cellists, and 5 pianists if each quartet is to consist of one player of each instrument?

Solution:
$$\begin{aligned}\frac{5^4 \times 4^4 \times 3^4 \times 2^4 \times 1^4}{5!} &= (5!)^3,\\ &= 1\,728\,000.\end{aligned}$$

(b) i. If $t = \tan\theta$, prove that

$$\tan 4\theta = \frac{4t(1-t^2)}{1-6t^2+t^4}.$$

Solution:
$$\begin{aligned}\text{L.H.S.} &= \frac{2 \times \tan 2\theta}{1-(\tan 2\theta)^2},\\ &= \frac{2 \times \frac{2t}{1-t^2}}{1-\left(\frac{2t}{1-t^2}\right)^2},\\ &= \frac{4t(1-t^2)}{(1-t^2)^2-4t^2},\\ &= \frac{4t(1-t^2)}{1-6t^2+t^4},\\ &= \text{R.H.S.}\end{aligned}$$

ii. If $\tan\theta\tan 4\theta = 1$, deduce that $5t^4 - 10t^2 + 1 = 0$.

Solution:
$$\begin{aligned}t \times \frac{4t(1-t^2)}{1-6t^2+t^4} &= 1,\\ 4t^2 - 4t^4 &= 1 - 6t^2 + t^4,\\ 5t^4 - 10t^2 + 1 &= 0.\end{aligned}$$

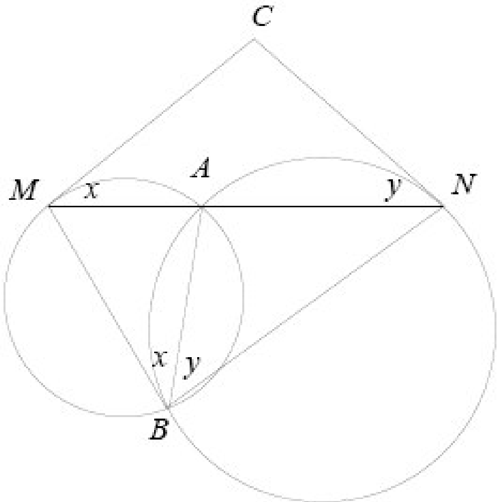
iii. Given that $\theta = \frac{\pi}{10}$ and $\theta = \frac{3\pi}{10}$ are roots of the equation $\tan\theta\tan 4\theta = 1$, find the exact value of $\tan\frac{\pi}{10}$.

Solution: Using the quadratic formula,
$$\begin{aligned}t^2 &= \frac{10 \pm \sqrt{100-20}}{10},\\ &= \frac{5 \pm 2\sqrt{5}}{5}.\end{aligned}$$

i.e., $t = \sqrt{\dfrac{5 \pm \sqrt{5}}{5}}$ as $\tan\frac{\pi}{10}, \tan\frac{3\pi}{10} > 0$.

Now, as $\tan\frac{\pi}{10} < \tan\frac{3\pi}{10}$, $\tan\frac{\pi}{10} = \sqrt{\dfrac{5-\sqrt{5}}{5}}$.

(c)



Two circles intersect at A and B. A line through A cuts the circles at M and N. The tangents at M and N intersect at C.

i. Prove that $\angle CMA + \angle CNA = \angle MBN$.

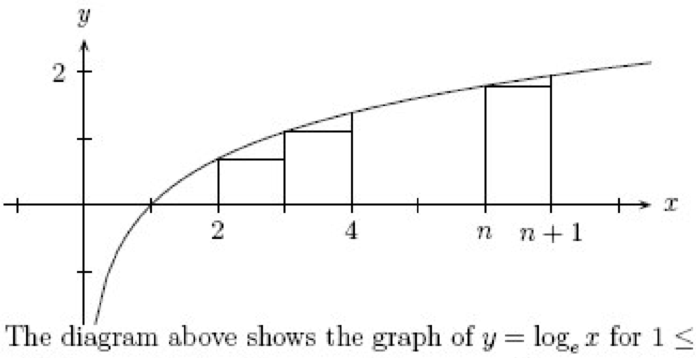
Solution: Join AB.

$$\begin{aligned} \angle CMA &= \angle MBA \text{ (angle in alternate segment)}, \\ \angle CNA &= \angle ABN \text{ (angle in alternate segment)}, \\ \therefore\ \angle CMA + \angle CNA &= \angle MBA + \angle ABN, \\ &= \angle MBN. \end{aligned}$$

ii. Prove M, C, N, B are concyclic.

Solution: $\angle CMA + \angle CNA + \angle MCN = 180°$ (angle sum of $\triangle CMN$),
$\therefore\ \angle MBN + \angle MCN = 180°$.
So $MCNB$ is a cyclic quadrilateral (opposite angles supplementary).

8. (a)



The diagram above shows the graph of $y = \log_e x$ for $1 \leq x \leq n + 1$.

i. By considering the sum of the areas of inner and outer rectangles, show that

$$\ln(n!) < \int_1^{n+1} \ln x \, dx < \ln((n+1)!)$$

Solution:
$$\begin{aligned}\text{Sum inner rectangles} &= \sum_{x=1}^{n} \ln x \times 1,\\ &= \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n,\\ &= \ln n!\\ \text{Sum outer rectangles} &= \sum_{x=2}^{n+1} \ln x \times 1, \text{ or } \sum_{x=1}^{n} \ln(x+1) \times 1,\\ &= \ln 2 + \ln 3 + \ln 4 + \cdots + \ln(n+1),\\ &= \ln(n+1)!\end{aligned}$$
$$\therefore \ \ln n! < \int_1^{n+1} \ln x \, dx < \ln(n+1)!$$

ii. Find $\int_1^{n+1} \ln x \, dx$.

Solution:
$$\begin{aligned} \text{I} &= \textstyle\int_1^{n+1} \ln x \times 1 \, dx, & u &= \ln x & v' &= 1\\ &= [x \ln x]_1^{n+1} - \textstyle\int_1^{n+1} dx, & u' &= \frac{1}{x} & v &= x\\ &= (n+1)\ln(n+1) - 0 - [x]_1^{n+1},\\ &= (n+1)\ln(n+1) - (n+1-1),\\ &= (n+1)\ln(n+1) - n.\end{aligned}$$

iii. Hence prove that
$$e^n > \frac{(n+1)^n}{n!}$$

Solution:
$$\begin{aligned}\text{From i., } \ln(n+1)! &> \textstyle\int_1^{n+1} \ln x \, dx.\\ \therefore \ \ln(n+1)! &> \ln(n+1)^{n+1} - n.\\ n &> \ln \frac{(n+1)^{n+1}}{(n+1)!},\\ &> \ln \frac{(n+1)^n}{n!}.\\ \therefore \ e^n &> \frac{(n+1)^n}{n!}.\end{aligned}$$

(b) If a root of the cubic equation $x^3 + bx^2 + cx + d = 0$ is equal to the reciprocal of another root, prove that
$$1 + bd = c + d^2.$$

Solution: Let the roots be α, $\frac{1}{\alpha}$, β.
Method 1:
$$\begin{aligned}\alpha \times \frac{1}{\alpha} \times \beta &= -d,\\ \beta &= -d.\end{aligned}$$
Substitute in the equation for the root β:

$$
\begin{aligned}
-d^3 + bd^2 - cd + d &= 0,\\
cd + d^3 &= bd^2 + d.\\
\text{Divide by } d \quad (d \neq 0),&\\
c + d^2 &= bd + 1.
\end{aligned}
$$

Method 2:

$$
\begin{aligned}
\alpha + \frac{1}{\alpha} + \beta &= -b,\\
1 + \alpha\beta + \frac{\beta}{\alpha} &= c,\\
\beta &= -d.\\
\therefore\ \alpha + \frac{1}{\alpha} - d &= -b \ \ldots \boxed{1}\\
1 - \alpha d - \frac{d}{\alpha} &= \quad c \ \ldots \boxed{2}\\
1 - c &= d(\alpha + \frac{1}{\alpha}),\\
\therefore\ \alpha + \frac{1}{\alpha} &= \frac{1-c}{d}.\\
\text{Sub. in } \boxed{1},\ \frac{1-c}{d} - d &= -b,\\
1 - c - d^2 &= -bd,\\
\textit{i.e.},\ 1 + bd &= c + d^2.
\end{aligned}
$$

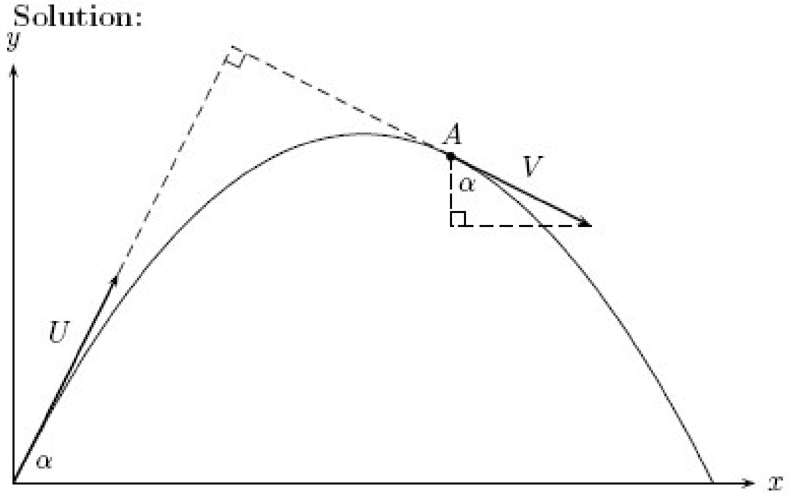
(c) A stone is projected from a point O on a horizontal plane at an angle of elevation α and with initial velocity U metres per second. The stone reaches a point A in its trajectory, and at that instant it is moving in a direction perpendicular to the angle of projection with speed V metres per second.

Air resistance is neglected throughout the motion and g is the acceleration due to gravity.

If t is the time in seconds at any instant, show that when the stone is at A:

i. $V = U \cot\alpha$

Solution:



$$\ddot{x} = 0 \qquad \ddot{y} = -g$$

$$\dot{x} = U\cos\alpha \qquad \dot{y} = U\sin\alpha - gt$$

At A, $U\cos\alpha = V\sin\alpha$,

$$i.e.,\ V = U\cot\alpha$$

ii. $t = \dfrac{U}{g\sin\alpha}$

Solution: At A, $\dot{y} = -V\cos\alpha$ (now heading downwards),

$i.e.,\ -U\cot\alpha \times \cos\alpha = U\sin\alpha - gt,$

$$\begin{aligned} gt &= U\sin\alpha + U\frac{\cos\alpha}{\sin\alpha}.\cos\alpha, \\ &= U\left(\frac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha}\right). \\ \therefore\ t &= \frac{U}{g\sin\alpha}. \end{aligned}$$